ON THE THEORY OF OPTIMUM SYSTEMS WITH AFTEREFFECT

(K TEORII OPTIMAL'NYKH SISTEM S POSLEDEISTVIEM)

PMM Vol.28, № 3, 1964, pp. 534-536

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(Received February 24, 1964)

Let us consider the system of integral-differential equations

$$L[z] \equiv \varepsilon \frac{dz}{dx} - A(x) z - \int_0^x K(x, t) z(t) \, dt = B(x)u \tag{1}$$

with the initial conditions $z(0) = b$ and the functional

$$J[u] = \int_0^\infty [z^*(t) C(t) z(t) + \int_0^t z^*(t) Q(t, s) z(s) \, ds + u^*(t) Hu] \, dt, \tag{2}$$

$$C(x) \geqslant 0, \quad Q(x, t) \geqslant 0, \qquad H(x) > 0, \quad \det A(x) \neq 0$$

where $0 < \varepsilon$ is a small parameter, $A(x)$, $K(x, t)$; $B(x)$, $C(x)$, $H(x)$ and $Q(x, t)$ are known matrix functions (C, Q and H are symmetric matrix functions), and b is an n-dimensional constant vector.

Let us pose the problem (D) of finding the control $U = U_\varepsilon(x)$ such that for $0 < \varepsilon < \varepsilon_0$ (where ε_0 is sufficiently small positive number), the equalities

$$\lim z_\varepsilon(x) = 0 \quad \text{for } x \to \infty, \qquad J[u] = \min \quad \text{for } \varepsilon \to 0$$

will be satisfied.

Let us consider the system of integral equations

$$L_1[v] \equiv A(x) v + \int_0^x K(x, t) v(t) \, dt = B(x) w(x) \tag{3}$$

and the functional

$$J[w] = \int_0^\infty \left[v^*(t) C(t) v(t) + \int_0^t v^*(t) Q(t, s) v(s) \, ds + w^* H w \right] dt \tag{4}$$

Let us pose the problem (D_0) of finding the control $w(x)$ such that the equalities

$$\lim_{x \to \infty} v(x) = 0, \qquad J[w] = \min$$

will be satisfied.

Henceforth we shall designate D_0 as degenerated with respect to the problem D, and the problem D to be perturbed with respect to the problem D_0.

Let $w_0 = w_0(x)$ be an admissible control for which

$$L_1[q] = B(x)\,w_0, \qquad l(x) = w_0(x) - w(x), \qquad p_1(x) = q(x) - v(x)$$

and let $R(x, s)$ be the resolvent of the kernel $A^{-1}(x)\,K(x, t)$. Then

$$L_1[p(x)] = B(x)\,l(x)$$

$$J[w_0] = J[w + l(x)] = J[w] + J[l(x)] +$$
$$+ 2\int_0^\infty \left[p^*(t)\,C(t)\,v(t) + \int_0^t p^*(t)\,Q(t, s)\,v(s)\,ds + l^*(t)\,H(t)\,w\right]dt \tag{5}$$

and the function

$$p(x) = \int_0^x R(x, s)\,A^{-1}(s)\,B(s)\,l(s)\,ds$$

satisfies the system $L_1[p] = B(x)\,l(x)$. We have

$$p^*(x) = \int_0^x l^*(s)\,R_0(x, s)\,ds$$

Here, $p^*(x)$, $l^*(x)$ are conjugate vectors relative to the vectors $p(x)$ and $l(x)$, and $R_0(x,s)$ is a known $n \times n$ matrix function.

Theorem 1. Let the system of integral equations

$$L_1[v] = B(x)\,a(x)$$

have the solution $v(x)$ such that

$$\lim_{x\to\infty} v(x) = 0, \qquad \int_0^\infty v^*(t)\,v(t)\,dt < +\infty$$

Then the function $w(x) = -H^{-1}(x)\,B^*(x)\,a(x)$ will be a unique optimum control where

$$a(x) = -B^{*-1}(x)\int_x^\infty R_0(t, x)\left[C(t)\,v(t) + \int_0^t Q(t, s)\,v(s)\,ds\right]dt \tag{6}$$

Proof. Let us assume now that the optimum control for the problem D_0 has the form

$$w(x) = -H^{-1}(x)\,B^*(x)\,a(x)$$

We have

$$\int_0^\infty \left[p^*(t)\,C(t)\,v(t) + \int_0^t p^*(t)\,Q(t, s)\,v(s)\,ds + l^*(t)\,H(t)\,w(t)\right]dt =$$
$$= \int_0^\infty \left\{\int_0^x l^*(s)\,R_0(t, s)\,ds\left[C(t)\,v(t) + \int_0^t \tilde{Q}(t, s)\,v(s)\,ds\right] + l^*(t)\,H(t)\,w(t)\right\}dt =$$
$$= \int_0^\infty l^*(s)\left\{\int_s^\infty R_0(t, s)\left[C(t)\,v + \int_0^t Q(t, s)\,v(s)\,ds\right]dt - B^*(s)\,a(s)\right\}ds \equiv 0$$

Then it follows from (5) and (6) that the $w(x)$ will be optimum control.

Let us say that the $n \times n$ matrix function $A(x)$ satisfies the condition (S) if for all $x \geqslant 0$ the inequality

$$\tfrac{1}{2}[A(x) + A^*(x)] < -\alpha E_m < 0 \qquad (\alpha = \text{const}, \alpha > 0)$$

is valid.

Lemma 1. If the matrix function $A(x)$ satisfies the condition (S) then for all $x \geqslant s \geqslant 0$ for the fundamental matrix

$$W_\varepsilon(x, s) \;\; [W_\varepsilon(s, s) = E_n]$$

of the system of Equations

$$\varepsilon y' = A(x) y$$

the inequality

$$\|W_\varepsilon(x, s)\| \leqslant K \exp\frac{-\alpha(x-s)}{\varepsilon} \qquad (K = \text{const})$$

is satisfied.

Let us consider the system $\varepsilon\pi'(x, \varepsilon) = A(x)\pi(x, \varepsilon)$ with the initial condition of the form $\pi(0) = b_1 - v(0)$. On the basis of Lemma 1, for all $x \geqslant 0$ the inequality

$$\|\pi(x, \varepsilon)\| \leqslant K_0 \exp\frac{-\alpha x}{2} \qquad (K_0 = \text{const})$$

will be valid for $\pi(x, \varepsilon)$.

By means of the substitution

$$z(x, \varepsilon) = v(x) + \pi(x, \varepsilon) + \varepsilon\xi(x, \varepsilon), \qquad u(x, \varepsilon) = w(x) + \varepsilon\eta(x, \varepsilon)$$

we reduce the system of equations (1) and the functional (2) to the form

$$L[\xi] = B(x)\eta(x) - v'(x) + \int_0^x K(x, t)\frac{1}{\varepsilon}\pi(t, \varepsilon)\,dt \tag{7}$$

$$\xi(0, \varepsilon) = 0$$

$$J[u] = J[w + \varepsilon\eta] = J[w] + \varepsilon J[\eta] + J_0(\varepsilon) + J_1(\varepsilon)$$

where

$$J_0(\varepsilon) \equiv 2\int_0^\infty \pi^*(t, \varepsilon)\left[C(t)\,v(t) + \int_0^t Q(t, s)\,\pi(s, \varepsilon)\,ds\right]dt$$

$$J_1(\varepsilon) \equiv 2\varepsilon\int_0^\infty [A_0(t)\,\xi(t) + w^*(t)\,H(t)\,\eta(t)]\,dt$$

$$A_0(t) \equiv v^*(t)\,C(t) + \int_0^t v^*(s)\,Q(t, s)\,ds + \pi^*(t)\,C(t) + \int_0^t \pi^*(t, \varepsilon)\,Q(t, s)\,ds \tag{8}$$

It is easy to see that for $\sup(\|C(x)\|\|v\|\|Q(x, \varepsilon)\| \leqslant C_0 = \text{const}$ the integral $I_0(\epsilon) \to 0$ as $\varepsilon \to 0$.

Let $V_\varepsilon(x, s)$ $[V_\varepsilon(s, s) = E_n]$ satisfy the system

$$\varepsilon\frac{dy}{dx} = A(x)\,y + \int_s^x K(x, t)\,y(t, s)\,ds, \qquad s \in [0, \infty)$$

where s is a certain parameter. Then the function

$$\xi(x, \varepsilon) = \int_0^x V_\varepsilon(x, s)\frac{1}{\varepsilon}B(s)\,\eta(s)\,ds - \int_0^x V_\varepsilon(x, s)\,\frac{1}{\varepsilon}\left[\int_0^s K(s, t)\,\pi dt - v'(s)\right]ds \tag{9}$$

satisfies the system (7) and the initial condition $\xi(0, \varepsilon) = 0$.

Let the function $\eta(x)$ satisfy the system

$$w^*(x) H(x) \eta(x) = A_0(x) \left\{ \int_0^x V_\varepsilon(x, s) \frac{1}{\varepsilon} B(s) \eta(s)\, ds - \right.$$

$$\left. - \int_0^x V_\varepsilon(x, s) \frac{1}{\varepsilon} \left[\int_0^s K(s, \nu) \pi(\nu, \varepsilon)\, d\nu - v'(s) \right] ds \right\} \qquad (10)$$

Then there results from (8) and (9) that $J_1(\varepsilon) = 0$. Thus the following theorem is proved.

T h e o r e m 2. If the system of equations (7) and (10) has the solutions $\xi(x, \varepsilon)$ and $\eta(x, \varepsilon)$ such that

$$\lim \xi(r, \varepsilon) = 0, \qquad \int_0^\infty \xi^*(t, s)\, \xi(t, \varepsilon)\, dt < +\infty$$

then for $\varepsilon \to 0$ the function $u(x) = w(x) + \varepsilon\eta(x, \varepsilon)$ will be a unique optimum control of the perturbed problem D

In conclusion, let us note that the idea of the present paper is applicable for more general systems with aftereffect [4].

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Translated by M.D.F.